# ECONOMIC GROWTH AND REDISTRIBUTION: EVIDENCE FROM DYNAMIC GAMES

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## ABSTRACT

This work is based on a differential game proposed by Kelvin Lancaster. The game between two agents called workers and capitalists is based on the accumulation and redistribution of benefits among social classes concluding that cooperative outcomes outperform non-cooperative. This approach determines the optimum solution in a centralized economy as a non-cooperative game and cooperative Pareto solutions forced by the social planner. We compare the model solution in a decentralized economy, where rates economic growth are converging to a steady state and obtain high rates of inflation, and higher levels of consumption. Alternatively, the cooperative solution found between agents can be confirmed. A capitalist game continues to monitor the cooperative principle, in which the maximum benefit is obtained through cooperation.

**JEL:** C61, C71, C72, C73

KEYWORDS: Differential game, non-cooperative game and cooperative game

## **INTRODUCTION**

Since the publication of the book of von Neumann and Morgenstern (1994) The Theory of Games and Behavior, many authors have relied on game theory to represent a vast range of dynamic conflict situations in the field of economic theory. We consider the work of Kelvin Lancaster (1973) as the first reference to the application of differential games to economic growth and redistribution in our modern economy. Lancaster made a differential game between two agents called workers and capitalists, based on the topics of accumulation and redistribution of benefits among social classes. He studied the classics such as Malthus, Ricardo and Marx, concluding that cooperative outcomes outperform non-cooperative.

The work of Lancaster, Hoel (1978) considered the utility function of players discounted consumption over time. On the other hand Pohjola (1983) compared the Nash solution found by Lancaster with the Stackelberg solution in open cycle. This article would be years later commented on by de Zeeuw (1992) showing that the solution found by Pohjola is not complete and that there are infinitely Stackleberg balances. Similarly, Soto and Ramos (1992), made the same comparison of results in a Stackelberg game solution, and a cooperative Pareto solution, modifying the model, where both players seek to maximize the present value of consumption, when the two updated at the same well, reaching the result that the cooperative solution is Pareto more efficient than the one found in Stackelberg solution.

Finally, we find the solutions of a differential game, with changes to the original approach of Lancaster. We compare the solutions found to the Nash and Paretian solution, under a centralized economy and solutions in a decentralized economy in an infinite time horizon, where two agents present value is discounted to their consumption at different rates. Then we study the dynamic behavior of the economic growth model proposed by modifying the golden rule posed by Ramsey (1928) to maximize the use of both agents over time constant and determining the value of redistribution that should be awarded in the economy for both agents.

The work is outlined as follows: Part two details the model, section three develops the model's economic growth under a centralized approach. Later in section four with a decentralized view. We finally draw some conclusions.

# LITERATURE REVIEW

As noted by Matti Pohjola, in his brief collection of papers on differential games of capitalism (Pohjola, 1985) the first approach to the problem of distribution from the point of view of game theory was proposed by Phelps and Pollask (1968). These authors see economic growth and redistribution as an intergenerational conflict. Kelvin Lancaster (1973) was first to apply game theory, differential patterns of growth and redistribution. He formulated a differential game of two players considering accumulation and redistribution of benefits among social classes.

Lancaster (1973) divided society into two classes called workers and capitalists, who are the game players. While his terminology is perhaps outdated, the work remains valid. Assume a sector economy whose output at certain instant can be consumed or added to existing capital stock. а Lancaster's model is designed to calculate the Nash equilibrium in open cycle (which happens to also balance feedback) and compares them with the cooperative solution that is obtained when both players seek jointly maximize the sum of As expected the results to consumption. show competition produces a deadweight loss.

Several authors have continued the study of capitalism games, introducing new hypotheses and developing new versions of the original model of Lancaster. Michael Hoel (1978) considered the effect of utility functions of players discounted consumption over time, which makes the equation governing dynamics of the system nonlinear. Pohjola (1985) introduced an infinite horizon, nonlinear utility functions and a restriction of full employment.

Some studies have addressed ways to reduce the welfare loss due to the lack of cooperation between players. Pohjola (1983b) shows that allowing workers to bear some control over investment improves the welfare of both players. Buhl and Machaczek (1985) along these lines consider the implications for workers having ownership of capital. This reduces the possible inefficiency as the decisions of workers are less tied to the behavior of capitalists. Moreover Pohjola (1983a) compares the Nash solution found by Lancaster with the Stackelberg solution in open cycle. This item was discussed later by Aart de Zeeuw (1992) showing that the solution found by Pohjola is not complete and that there are infinite Stackelberg equilibrium, but the findings on consumption remain true.

Cooperative solutions have also been studied in this context. Comparing Lancaster to the Nash equilibrium is the limit of what he calls social welfare function, which is simply the sum of consumption of both players. Hoel (1978) does the same but considers the amount instead of the entire Pareto frontier of all possible payments.

Rincon (1994) and Soto (1994) considered a modified model in which capitalists control the supply of labor, in addition to investment, and where the production function is Cobb-Douglas type. Soto noted threat strategies are not efficient but are а perfect balance in the subgames. Seierstad (1993) defined trigger strategies and demonstrates two different types. In addition to being perfect for subgames, payments more efficient this result obtains than Nash. Here we describe a model of growth and redistribution, formalizing the dynamics of the system and the functions of paying players.

## THE MODEL

Considered a closed economy, the production Y(t); en in an instant of time *t* can be either consumed or intended for investment capital stock K(t). In this sector economy are two social agents called "working consumers" and "consumer investors," we assume the production function takes the form proposed by Rebelo (1991):  $Y(t) = f(K) = A \cdot K(t)$ , where A is the technology parameter. Additionally, some assumptions are adopted from the model proposed by Ramsey (1928) namely the infinite life that pose individuals, this due to the existence of dynasties. Also, we note that the population grows at a rate "n" and there is no physical depreciation of capital or Y(t) is the net rather than gross production.

At each instant of time "working consumers" can be controlled with a variable proportion of total output that is targeted towards consumption. Following González (1998) we assume that this variable is defined in an interval  $[a,b] \subseteq [0,1]$ . The consumption of workers will be denoted by  $C_1(t)$  and is given by:  $C_1(t) = \alpha(t) \cdot \theta \cdot f[K(t)]$ , where " $\theta$ " is the marginal propensity to consume.

Consumers investors have at their disposal all the output not consumed by the working consumers. It is intended either for their own consumption or to reinvest in the economic system. With the variable  $\beta(t) \in [c,d] \subseteq [0,1]$  control the proportion devoted to investment. The rest  $C_2(t)$ , their consumption is  $C_2(t) = [1 - \alpha(t)] \cdot [1 - \beta(t)] \cdot \theta \cdot f[K(t)]$ . Investment in this economy is given by the variation in the accumulation of capital stock. This leads to the following differential equation:

$$\overset{\bullet}{\mathbf{K}(t)} = \mathbf{I}(t) = \begin{bmatrix} 1 - \alpha(t) \end{bmatrix} \cdot \boldsymbol{\beta}(t) \cdot \boldsymbol{\theta} \cdot \mathbf{f} \begin{bmatrix} \mathbf{K}(t) \end{bmatrix}$$
(1)

Players seek to maximize their utility over an infinite time horizon, unlike the model proposed by Lancaster (1973), Pohjola (1983), Hoel (1978), Soto and Fernandez (1992) and Gonzalez (1998) who work with infinite horizon. Following Hoel, the utility we identify with the consumer using a positive discount factor denoted as  $\lambda$  for "working consumers" and  $\psi$  to "consumers investors". Thus we have the payoff function for each of the players:

$$J_{1} = \int_{0}^{+\infty} \alpha(t) \cdot \theta \cdot f[K(t)] e^{-\lambda t} \cdot dt \quad (2) \quad \text{and} \quad J_{2} = \int_{0}^{+\infty} [1 - \alpha(t)] \cdot [1 - \beta(t)] \cdot \theta \cdot f[K(t)] e^{-\psi t} \cdot dt \quad (3)$$

It only remains to establish a state of the capital stock at the initial instant:

$$\mathbf{K}(0) = \mathbf{K}_0 \tag{4}$$

The game consists of finding for each player as a watchdog over time varying in [a,b] and [c,d] respectively for each player, so as to maximize (2) and (3) subject to system dynamics described by (1) as an initial condition given by (4).

## IMPLICATIONS OF A CENTRALIZED ECONOMY

In this economy there is a benevolent social planner that seeks to maximize the welfare of society as measured by the level that they consume.

#### Non-cooperative Solution: Nash Equilibrium

The calculation of Nash equilibrium is the maximization of the objective functional measures seeking to maximize the payment of a player while the others are considered fixed. In this situation hamiltoneanos define two for each player respectively, so we have "working consumers"

$$H_{1}(t, K(t), \alpha(t), \beta(t), m_{1}(t)) = \alpha(t) \cdot \theta \cdot f[K(t)] \cdot e^{-\lambda t} + m_{1}(t)[[1 - \alpha(t)]] \cdot \beta(t) \cdot \theta \cdot f[K(t)]]$$
(5)

And "consumer's investors":

$$H_{2}(t, K(t), \alpha(t), \beta(t), m_{2}(t)) = [1 - \alpha(t)] \cdot [1 - \beta(t)] \cdot \theta \cdot f[K(t)] \cdot e^{-\psi t} + m_{2}(t) [[1 - \alpha(t)] \cdot \beta(t) \cdot \theta \cdot f[K(t)]]$$

$$(6)$$

The solution of this optimal control problem is solved based on the principle of maximum Pontrygain, taking into account that  $\dot{K(t)} = (1 - \alpha(t))\beta(t)\theta f[K(t)]$  and  $K(0) = K_0$ ; thus optimal controls are:  $(\alpha^*(t) = b \ y \ \beta^*(t) = c)$ .

Being-variables problem are:

At time " $\vec{t}$ " which is left to use the combination  $(\alpha^*(t) = b \ y \ \beta^*(t) = c)$ , is one in which cease verified the optimality conditions of the pair of control. The solution is obtained by solving the "t" equations  $e^{-\lambda t} = m_1(t)\beta(t)$  and  $m_2(t) = e^{-\psi t}$ . In which  $t_1$  and  $t_2$  are the respective solutions of the equations above, if they exist and are in the interval  $[0, +\infty)$ , otherwise, give zero. It is noteworthy that for the resolution of equations that will approximate the equations by Taylor's estate.

$$\begin{split} t_1 &\approx \frac{A \cdot \theta \cdot (b \cdot (c^2 - c \cdot Q_1 \cdot -1) - c \cdot (c - Q_1)) + \lambda \cdot (c - Q_1)}{A^2 \cdot c^2 \cdot Q_1 \cdot \theta^2 \cdot (b - 1)^2 + A \cdot \theta \cdot \lambda \cdot (b \cdot (c^2 + c \cdot Q_1 - 1) - c \cdot (c + Q_1)) + c \cdot \lambda^2} \\ t_2 &\approx \frac{A \cdot c \cdot \theta \cdot (b - 1) \cdot (Q_2 - 1) - b \cdot \theta \cdot (c - 1) + c \cdot \theta - Q_2 \cdot \psi - r + \psi}{A^2 \cdot c^2 \cdot Q_2 \cdot \theta^2 \cdot (b - 1)^2 - A \cdot c \cdot \theta \cdot \psi \cdot (b - 1) \cdot (Q_2 - 1) + \psi \cdot (b \cdot \theta \cdot (c - 1) + c \cdot \theta - \theta + \psi)} \end{split}$$

For the value of optimal control pair, the value of the stock of capital in the economy will:

$$K(t) = K_0 \cdot e^{\left[\theta \cdot A \cdot (1-b) \cdot c\right]t}$$
(8)

#### The Transversality Condition

Given the condition  $\lim_{t\to\infty} (m_t \cdot k_t) = 0$ , then replace the values of (7) and (8), to clear the values of the constants  $Q_1$  and  $Q_2$ ;

$$\lim_{t\to\infty} \left( \left[ \frac{b \cdot \theta \cdot A \cdot e^{\left[\theta \cdot A \cdot (1-b) \cdot c - \lambda\right]t}}{\lambda - A \cdot \theta \cdot (1-b) \cdot c} + Q_1 \right] \cdot \left[ K_0 \cdot \right] \right) = 0 \text{ And}$$

$$\lim_{t \to \infty} \left( \left[ \frac{(1-c) \cdot (1-b) \cdot \theta \cdot A \cdot e^{\left[\theta \cdot A \cdot (1-b) \cdot c - \psi\right]t}}{\psi - A \cdot \theta \cdot (1-b) \cdot c} + Q_2 \right] \cdot \left[ K_0 \cdot \right] \right) = 0$$

It is important to note, as a necessary condition that  $[\theta A(1-b)c - \lambda] < 0$  and  $[\theta A(1-b)c - \psi] < 0$ , which requires that the value of  $Q_1$  and  $Q_2$ , are zero.

#### Value of Redistribution

The value of redistribution that should continue throughout the time the two actors in this economic system, "consumers workers" and "consumers investors" should be the payoff function expressed in (2) and (3). Then the value to be redistributed in the economy for both agents in order to maximize their utility as expressed in its use shall be:

$$\begin{cases} J_1 = \frac{b\theta AK_0}{\lambda - \theta A(1 - b)c} \\ \\ J_2 = \frac{(1 - c)(1 - b)\theta AK_0}{\psi - \theta A(1 - b)c} \end{cases}$$
(9)

Deriving the optimal objective function with respect to K<sub>0</sub>, we obtain:

$$\begin{cases} \frac{\partial J_1}{\partial K_0} = \frac{b\theta A}{\lambda - \theta A(1 - b)c} \\ \frac{\partial J_2}{\partial K_0} = \frac{(1 - c)(1 - b)\theta A}{\psi - \theta A(1 - b)c} \end{cases}$$

It can be seen in the previous derivative experts we give interpretations of the variable price Coester. The product  $m_1 e^{\lambda t}$  measures approximately the total income increase optimal for player 1, by increasing the initial request on the stock of capital by one unit. Similarly, the product  $m_2 e^{\psi t}$  measures the increase in optimal total profit of player 2, by increasing the initial request on the stock of capital in one unit.

#### Cooperative Solution: Pareto Optimal

Now we will find non-dominated elements of the set of all possible payments in the game described above. In which the social planner, implanted the cooperative state, for this we define a social welfare

function in which payments are in addition to giving players for  $J_1$  and  $J_2$ , weighted according to their ability or power of decision:

Max. 
$$W = \lambda J_{1} + (1 - \lambda) J_{2}$$
  
s.a.:  
$$\alpha(t) \in [a, b], \quad \beta(t) \in [0, 1] \quad \forall t \in [0, +\infty)$$
(8a)  
$$K(t) = (1 - \alpha(t)) \beta(t) \theta f[K(t)]$$
$$K(0) = K_{0}$$

Hamiltoneano then the optimal control problem is:

$$H(t, K(t), \alpha(t), \beta(t), m(t)) = \left[ \lambda \alpha(t) e^{-\delta t} + (1 - \lambda)(1 - \alpha(t))(1 - \beta(t)) e^{-\delta t} \right] \theta f[K(t)] + m(t)(1 - \alpha(t))\beta(t)\theta f[K(t)] = e^{-\delta t} \theta f[K(t)] \left\{ \lambda \alpha(t) + \left[ (1 - \lambda)(1 - \beta(t)) + m(t) e^{\delta t} \beta(t) \right] (1 - \alpha(t)) \right\}$$
(9b)

Plausible solutions in optimal control to be used highlight the possible combination of  $(\alpha(t) = b, \beta(t) = 1)$ and  $(\alpha(t) = b, \beta(t) = 0)$ , which will be used as times to take the social planner, called for "Maximum accumulation" and "Maximum Consumption" respectively. Thus, the peak consumption period, is interpreted as the ideal period to stimulate the economy, while the maximum accumulation period is the period to be analyzed if it seeks to anticipate any crisis that affects the growth of the economy. For these two periods there, redistribution of consumption values for both agents, maximizing their use. Maximum consumption is:

$$\begin{cases} J_1 = \frac{b\theta AK_0}{\lambda} \\ J_2 = \frac{(1-b)\theta AK_0}{\phi} \end{cases}$$

While for the period of "maximum accumulation" will have to consider that  $[\theta A(1-b) - \lambda] < 0$ , then the value will be redistribution to agents:

$$\begin{cases} J_1 = \frac{b\theta AK_0}{\lambda - \theta A(1 - b)} \\ J_2 = 0 \end{cases}$$

## Economic Growth

In this economy (assuming that the social planner takes the non-cooperation) is important to note that the rate of growth of per capita production will coincide with the growth rate of capital stock per capita and consumption per capita terms.

$$\gamma_{v} = \gamma_{k} = \gamma_{\overline{c}} = \theta \cdot A \cdot (1 - b) \cdot c - n \tag{10}$$

However, the analysis of the law of evolution of capital per capita, leads us to three separate cases, in which two of them invite us to a possible steady state.

$$\mathbf{\hat{k}} = \begin{bmatrix} \theta A(1-b)c - n \end{bmatrix} \mathbf{\hat{k}} = 0$$

$$\begin{cases} Caso I : \mathbf{k} = 0 \\ b \end{pmatrix} \begin{bmatrix} a \theta A(1-b)c > n \\ b \end{pmatrix} \theta A(1-b)c < n \\ Caso II : \mathbf{k} \neq 0 \\ c \end{pmatrix} \begin{bmatrix} c \theta A(1-b)c = n \end{bmatrix}$$

$$(11)$$

The three cases shown in (11) have different economic interpretations. Case I refers to the extreme state of an economy that has no capital stock or sold out to generate resources, which can generate two hypothetical subcases; a) An economy in which  $\theta A(1-b)c > n$ , then the law of evolution of capital stock, generates economic growth. Then an economy will be imposed as a prerequisite for generating economic growth, the implementation of this case. Being as amended the new golden rule of capital stock, to the simplicity of a difference, to be verified over time.

$$(1-s)A(1-b)c - n > 0$$
$$s < 1 - \frac{n}{A(1-b)c}$$

b) An economy in which  $\theta A(1-b)c < n$ , implying that the rate of economic growth would be decreasing over time.

Case II is an economy where the population growth rate is equal to the value of  $\theta A(1-b)c$ , and given that  $\theta = 1-s$  where s is the marginal propensity to save in this economy, savings policy has led to a savings rate that does not generate economic growth:

$$(1-s)A(1-b)c - n = 0$$
$$s = 1 - \frac{n}{A(1-b)c}$$

Graphic evolution of the law of capital stock is represented in Figure 1.

Figure 1: Law of Evolution of per Capita Capital Stock



This figure shows the law of evolution of per capita capital stock

Returning to the economic growth rates, according to Figure 1, the use of each case in the economy, results in different rates of growth in capital stock and output per capita. In case I, there is sub special cases of much economic interest. Therefore we build on the cooperative solution prompted by a social planner, and show the behavior of the economy from the period of "maximum accumulation" and " Maximum Consumption" The graphical depiction of this situation is presented in Figure 2.

Figure 2: Dynamics of Consumption and Capital (Phase Diagram)



This figure shows dynamics of consumption and capital (phase diagram)

In an economy, "Maximum accumulation", where the growth rate will be  $\gamma_k = (1-b)\theta A - n$ , assuming it meets the new golden rule, this growth rate will always be increasing over time, instead of approximating the value of the marginal propensity to save the value of  $1 - \frac{n}{A(1-b)c}$ , and the growth rate of capital stock

will decrease continuously over time. Finally under a policy of "Maximum Consumption" the rate of economic growth will be negative,  $\gamma_k = -n$ . Met with the new golden rule, the dynamics of the model indicates that with increasing capital stock aggregate consumption is maximized.

The dynamics of the system shows a saddle point in which the only point of equilibrium is the origin, and solutions are the hyperbolic model, which shows that over time the consumption permanently grows which increases the capital stock in aggregate.

## The Convergence of the Economies

To analyze convergence between countries, suppose there is a set of countries for which the parameters  $\theta$ , A, b, c have the same values. If we assume that the only difference between them is the initial value of the stock of capital per person, k (0), then the model predicts that the growth rate of countries considered being constant and equal to  $\gamma_y = \gamma_k = \gamma_{\overline{c}} = \theta \cdot A \cdot (1-b) \cdot c - n$ . This model therefore predicts no convergence between the economies, absolute or conditional, since the growth rate is not related to income.

Suppose now that countries differ in the productivity parameter denoting, "A". Since the rate of growth of economies is given by equation (19), countries with high growth rates continue indefinitely, while those countries with low growth will remain forever with reduced growth rate regardless of income or value of its initial product.

## IMPLICATIONS IN A DECENTRALIZED ECONOMY

We now proceed to find the optimal redistribution when there is no presence of a social planner who cares about the welfare of economic agents. As "economic agents", are called upon to find the values of decision  $(k_t)$  and  $(c_t)$ . We assume there are two factor markets (production inputs), one for labor services,

wages, denoted by "<sup>w</sup>t", "<sup>r</sup>t" is the rental price of capital. Based on the model introduced by Sidrauski (1967), in addition the Central Bank prints fiat money (with zero cost) and real assets in return get back to families in the form of a transfer S. Then, we define the variable "M" as nominal balances, "p" price index

and "m" (M / p) real balances, and finally introduce the variable " $\pi = \frac{p}{p}$ " the rate of inflation in the

economy. In real terms and per capita budget constraint of the representative household is given by:

$$w + rk + S = c + \frac{k}{L} + \frac{M}{p}$$

Where  $\frac{M}{p}$  represents the "investment" in nominal balances expressed in real terms. Note that:

 $k = \frac{K}{L} \Rightarrow k = \frac{K}{L} - kn$  and  $m = \frac{M}{p} \Rightarrow m = \frac{M}{p} - \pi m$ . Therefore the budget constraint of the representative

family is amended:

$$w + rk + S = c + k + kn + m + \pi m$$
(12)

It is clear from (12) that real balances can be interpreted as an asset that depreciates at a rate equal to inflation  $\pi$ . The problem that the representative family, according to the consumption changes, is considered that of "worker" or "capitalist." Assuming that preference is given to consume the "worker" and then the "capitalist" maximizing the problems are:

$$J_{1} = \int_{0}^{+\infty} (t) \cdot \theta \cdot f[k(t)] e^{-\lambda t} \cdot dt$$
(13)

$$J_{2} = \int_{0}^{+\infty} (1 - \alpha(t))(1 - \beta(t)) \cdot \theta \cdot f[k(t)] e^{-\psi t} \cdot dt$$
s.a.:
$$(14)$$

$$\overset{\bullet}{\mathbf{k}} = \mathbf{w} + (\mathbf{r} - \mathbf{n})\mathbf{k} + \mathbf{S} - \alpha(\mathbf{t}) \cdot \mathbf{\theta} \cdot \mathbf{f} [\mathbf{k}(\mathbf{t})] - \begin{pmatrix} \bullet \\ \mathbf{m} + \pi \mathbf{m} \end{pmatrix}$$
$$\overset{\bullet}{\mathbf{m}} = \mathbf{\Phi}$$

The control variable is  $\alpha(t)$ ,  $\beta(t)$  and  $\phi$ , the state variables are k and m. to express the problem in standard form defines an auxiliary control variable  $\phi$ , so that  $\mathbf{m} = \phi$ . In this situation hamiltoneanos define two for each player respectively, so we have two "consumer's workers".

$$H_{1}(t, K(t), \alpha(t), \beta(t), \mu(t)) = \alpha(t) \cdot \theta \cdot f[K(t)] \cdot e^{-\lambda t} + \mu_{1}(t) \left[ w + (r - n)k + S - \alpha(t) \cdot \theta \cdot f[k(t)] - \begin{pmatrix} \bullet \\ m + \pi m \end{pmatrix} \right]$$

$$+ \mu_{2}(t) [\phi]$$
(15)

And the "consumer's investors":

$$H_{2}(t, K(t), \alpha(t), \beta(t), \upsilon(t)) = [1 - \alpha(t)] \cdot [1 - \beta(t)] \cdot \theta \cdot f[K(t)] \cdot e^{-\psi t} + \upsilon_{1}(t) \left[ w + (r - n)k + S - \alpha(t) \cdot \theta \cdot f[k(t)] - \begin{pmatrix} \bullet \\ m + \pi m \end{pmatrix} \right] + \upsilon_{2}(t) [\phi]$$

$$(16)$$

Applying the conditions of the principle of Maximum Pontrygain, for a couple of controls are optimal  $(\alpha^*(t), \beta^*(t))$ , we obtain:

$$\lambda = \pi \tag{17}$$

Solving for the "consumer investor", we have:

$$\Psi = \pi \tag{18}$$

So we can say, equating (17) and (18), the value of the intertemporal discount factor for both players coincide with the rate of inflation in the economy, which in a decentralized economy, players have adaptive expectations, weighted with which to change their present consumption for future consumption, according to the perception of the rate of inflation. Thus for perceptions of a high inflation rates, players require more current consumption in exchange for future consumption and on the contrary, perceptions of a lower interest rate, players will be willing to give higher current consumption.

The government's problem is simply to satisfy his budget constraint, assuming that transfers all proceeds from the creation of money. Then the following equation must be fulfilled:

$$\mathbf{S} = \mathbf{m} + \pi \mathbf{m} \tag{19}$$

Where S is the transfer given to the representative household. Thus, the budget constraint of the aggregate economy is obtained by substituting (19) into (12) and becomes:

 $\mathbf{k} = \mathbf{w} + (\mathbf{r} - \mathbf{n})\mathbf{k} - \mathbf{c}$ 

Therefore, the condition of the maximum principle, for the player "consumer worker":

$$\mu_{1} = \frac{\left[\theta\alpha(t)A + \pi\right]e^{-\pi t}}{\left[\theta\alpha(t)A - (r - n)\right]}$$

And the value of capital stock per capita, given a  $\alpha$  will be:

$$k(t) = \frac{w}{(r - n - A\theta\alpha)} + (k_0 - k^*)e^{[A\theta\alpha + n - r]t}$$

And by the transversality condition, we have:

$$\lim_{t\to\infty}\left\{\frac{\left[\theta\alpha(t)A+\pi\right]\!e^{-\pi t}}{\left[\theta\alpha(t)A-(r-n)\right]}\cdot\left(\frac{w}{\left(r-n-A\theta\alpha\right)}+\left(k_{0}-k^{*}\right)\!e^{\left[A\theta\alpha+n-r\right]\!t}\right)\right\}=0$$

Which deduced that the value of  $\alpha(t)$ , will be the minimum value that can be taken so that the transversality condition is not altered. Therefore, the optimal value of  $\alpha(t)$  is  $\alpha^*(t) = a$  Similarly, for the player "consumer investor, the value of the variable being-be:

$$\upsilon_1 = \frac{\left[\theta(1-\alpha(t))(1-\beta(t))A + \pi\right]e^{-\pi t}}{\left[\theta(1-\alpha(t))(1-\beta(t))A - (r-n)\right]}$$

And the value of capital stock per capita, given  $\alpha$  and  $\beta$ , will be:

$$k(t) = \frac{W}{\left(r - n - \left(1 - \alpha(t)\right)\left(1 - \beta(t)\right)A\theta\right)} + \left(k_0 - k^*\right)e^{\left[A\theta\left(1 - \alpha(t)\right)\left(1 - \beta(t)\right) + n - r\right]t}$$

And by the transversality condition, we have:

$$\lim_{t\to\infty}\left\{\frac{\left[\theta(1-\alpha(t))(1-\beta(t))A+\pi\right]e^{-\pi t}}{\left[\theta(1-\alpha(t))(1-\beta(t))A-(r-n)\right]}\cdot\left(\frac{w}{(r-n-(1-\alpha(t))(1-\beta(t))A\theta)}+\left(k_0-k^*\right)e^{\left[A\theta(1-\alpha(t))(1-\beta(t))+n-r\right]t}\right)\right\}=0$$

We deduced that the value of  $\beta(t)$ , will be the maximum value that can be taken so that the transversality condition is not altered. Therefore, the optimal value of  $\beta(t)$  is  $\beta^*(t) = d$ . It is easy to realize that in this decentralized economy, we find the presence of steady state capital stock in per capita consumption. Likewise, and this because the long term, the per capita stock of capital that can generate "economic agents", is limited.

The capital stock steady state "consumer worker" is:

$$k^{*}(t) = \frac{w}{(r - n - A\theta\alpha)}$$
(20)

The capital stock steady state "consumer investor" is:

$$k^{*}(t) = \frac{w}{(r - n - (1 - \alpha(t))(1 - \beta(t))A\theta)}$$
(21)

In both cases, we observe that there is a steady state with economic sense. We must consider as a necessary condition that  $r > n - (1 - \alpha(t))(1 - \beta(t))A\theta$ , so that increases in population growth result in higher interest rates in the economy.

#### Dynamics of the Decentralized Model

To study the dynamics of the model, we consider the dynamics governing the player "consumer worker". The dynamics of the player "consumer investor" is similar. To understand the dynamics of the model using the phase diagram of Figure 3, drawn on the plane  $(k_t, c_t)$ . All points of the first quadrant are feasible, except the points on the vertical axis above the origin, with no capital  $(if(k_t = 0))$ , production is zero, and therefore "c" positive is not feasible.

The locus of points  $(k_t, c_t)$  that satisfies  $\begin{pmatrix} \mathbf{\dot{k}} = 0 \end{pmatrix}$  starts from the value of " $w_t$ " and does not reach a maximum value of the stock of capital per capita. The locus satisfying  $\begin{pmatrix} \mathbf{\dot{c}} = 0 \end{pmatrix}$  is vertical to the stock of capital and is given as the steady-state value of capital stock. At any point above the locus  $\begin{pmatrix} \mathbf{\dot{k}} = 0 \end{pmatrix}$  the per capita capital is decreasing, the consumption is above the level that would keep constant. Similarly,  $k_t$  increases below the locus points  $\begin{pmatrix} \mathbf{\dot{k}} = 0 \end{pmatrix}$ . In the case of the locus  $\begin{pmatrix} \mathbf{\dot{c}} = 0 \end{pmatrix}$ , consumption increases to the left of that locus where  $(k_t = k^*)$  and decreases to the right of the locus.

Figure 3: Dynamics of Consumption and Capital (Phase Diagram)



This table shows the dynamics of consumption and capital.

Only the trajectory started at a level of capital stock of " $k_0$ " and " $c_0$ " that will lead us along the time path that converges to the point "E" will be the optimal trajectory which must be met for the economy to converge to the steady state or that the transversality condition is satisfied.

## Local Behavior around the Steady State

The liberalization of dynamics of the system generates other ideas on the dynamic behavior of the economy.

Linearizing the system around the steady state:

$$\begin{cases} \bullet \\ k = w + (r - n)k - c = \phi_1 \\ \bullet \\ c = \theta a A (A \theta a + n - r) (k_t - k^*) = \phi_2 \end{cases}$$

By Taylor's theorem and neglecting the terms with derivatives of order greater than one.

$$\begin{split} \varphi_{1}(k,c) &= \varphi_{1}(k^{*},c^{*}) + \frac{\partial \varphi_{1}(k,c)}{\partial k} \left(k - k^{*}\right) + \frac{\partial \varphi_{1}(k,c)}{\partial c} \left(c - c^{*}\right) \\ \varphi_{2}(k,c) &= \varphi_{2}(k^{*},c^{*}) + \frac{\partial \varphi_{2}(k,c)}{\partial k} \left(k - k^{*}\right) + \frac{\partial \varphi_{2}(k,c)}{\partial c} \left(c - c^{*}\right) \end{split}$$

Considering the steady state:

$$\begin{cases} \mathbf{e}_{1}^{\bullet}(\mathbf{t}) \equiv \phi_{1}\left(\mathbf{k}^{*}, \mathbf{c}^{*}\right) = \mathbf{0} \\ \mathbf{e}_{1}^{\bullet}(\mathbf{t}) \equiv \phi_{2}\left(\mathbf{k}^{*}, \mathbf{c}^{*}\right) = \mathbf{0} \end{cases}$$

The linearization of the laws of motion in the vicinity of the steady state gives:

$$\begin{bmatrix} \widetilde{k} \\ \widetilde{k} \\ \widetilde{c} \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -1 \\ \delta & 0 \end{bmatrix}}_{J} \begin{bmatrix} \widetilde{k} \\ \widetilde{c} \end{bmatrix}$$
(22)
Where:

Where:

The system (22) has the following solutions:

$$\widetilde{\mathbf{k}} = \mathbf{a}_1 \mathbf{e}^{\lambda_1 t} + \mathbf{a}_2 \mathbf{e}^{\lambda_2 t}$$

$$\widetilde{\mathbf{c}} = \mathbf{b}_1 \mathbf{e}^{\lambda_1 t} + \mathbf{b}_2 \mathbf{e}^{\lambda_2 t}$$
(23)

Where  $\lambda_1$  and  $\lambda_2$  are the characteristic roots or "eigenvalues" of the Jacobian matrix and  $a_1, a_2, b_1$  and  $b_2$  are constants to be determined. Calculation of eigenvalues:

$$\lambda_1 = \frac{\gamma + \sqrt{\Delta}}{2}$$
 And  $\lambda_2 = \frac{\gamma - \sqrt{\Delta}}{2}$ 

Where  $\Delta = \lambda^2 - 4\delta$  is the discriminant, and  $as \delta < 0 \Rightarrow \Delta > 0$ , so the roots are real and distinct characteristics. In addition  $\lambda_1 > 0$  and  $\lambda_2 < 0$ .

For the system formed by (23) converges to the steady state "E" we must remove him explosive effect  $\lambda_1$ . As we will impose  $a_1 = b_1 = 0$  and taking as initial conditions " $k_0$ " and " $c_0$ ":

$$k(t) = a_2 e^{\lambda_2 t} + k^* \implies k(0) = a_2 + k^* = k_0 \implies k(t) = (k_0 - k^*) e^{\lambda_2 t} + k^*$$
  

$$c(t) = b_2 e^{\lambda_2 t} + c^* \implies c(0) = b_2 + c^* = c_0 \implies c(t) = (c_0 - c^*) e^{\lambda_2 t} + c^*$$

Equating the value of  $e^{\lambda_2 t}$  and clearing c(t), we obtain:

$$c(t) = \frac{(c_0 - c^*)}{(k_0 - k^*)} (k(t) - k^*) + c^*$$
(24)

Equation (24) represents the linear approximation of the optimal path around "E". The convergence rate is given by  $|\lambda_2|$ . In this case  $|\lambda_2|$  is an increasing function of "r" (under perfect competition equals the value of (f'(k) = A)) and a decreasing function of "n". The higher the value of the interest rate of the economy, or equivalently, the higher the level of technology, the faster capital accumulation and the economy will converge to the steady state. However, higher levels of population growth, slower transition of the economy towards the steady state.

### Economic Growth of Decentralized Economy

In this economy it is important to note that the rate of growth of per capita production will coincide with the growth rate of capital stock per capita and consumption per capita terms.

$$\gamma_{v} = \gamma_{k} = \gamma_{\overline{c}} = (w - c)k^{-1} + (r - n)$$
<sup>(25)</sup>

The growth rate of the decentralized economy tends to be positive with higher wage levels per capita consumption per player, and higher interest rates to population growth.

## Redistribution among Agents

For this decentralized economy, redistribution of consumption values for both agents, maximizing their consumption is:

$$\begin{cases} J_1 = \frac{a\theta Aw}{\left[r-n-Aa\theta\right]\pi} + \frac{\left(k_0 - k^*\right)a\theta A}{\left[r+\pi-n-Aa\theta\right]} \\ \\ J_2 = \frac{\left(1-a\right)\left(1-b\right)\theta Aw}{\left[r-n-A\left(1-a\right)\left(1-b\right)\theta\right]\pi} + \frac{\left(k_0 - k^*\right)\left(1-a\right)\left(1-b\right)\theta A}{\left[r+\pi-n-A\left(1-a\right)\left(1-b\right)\theta\right]} \end{cases}$$

It should be noted that the payment received (redistribution of consumption) in the decentralized economy is greater than the payment received in a centralized economy (whether cooperative or not cooperative planned), this is explained because in this economy the inflation rate are each increasing with the increase in the level of wages, and thereby increases the intertemporal discount factor, so that agents require more current consumption for future consumption.

#### **Economic Policy**

Given the equation (17) and (18), in which we get  $\psi = \lambda = \pi$ , we can say that inflation is a function that depends on the consumption of agents. Econometric purposes, we propose the following equation for the behavior of inflation

$$\ln\left|\pi\right| = \beta_0 + \beta_1 \ln\left|C_t\right| \tag{26}$$

which can be rewritten as  $\ln|\pi| = \ln|K| + \beta_1 \ln|C_t|$ , where  $K = e^{\beta_0}$ . Therefore an infinitesimal change in  $\partial \pi$ 

consumption will result in changes in inflation, given a level of consumption,  $\frac{\partial \pi}{\partial C_t} = e^{\beta_0} \beta_1 C_t^{\beta_1 - 1}$ .

Equation (26) shows that with a consumption level higher than one, the level of inflation will be higher, while a consumption level of less than one, the economy will have deflation. It is also expected that the sign of  $\beta_0$  is negative, and the value of  $\beta_1$  is positive and known as the intertemporal setting the level of inflation.

On the other hand, when we speak of economic growth in the decentralized economy, there is no apparent steady state, but if the population growth rate exceeds the interest rate of the economy (n > r), the economy reaches a steady state, with levels of consumption and capital stock of static ( $c^*$ ,  $k^*$ ). To extend the steady state in time, and increase economic growth, can adopt two policies: a) Reduce "n" or permanently increase the value of "r" or b) Increase the level of wages "w", so as to obtain a sequence in time with wage levels increased { $w_a > w_b > w_c > ....w_{\infty}$ } and thus increase the value of capital stock steady state shown in (20) and (21), thus increasing the value of steady-state consumption. Given this policy (24), the level of consumption over time is increasing, and considering (26), the short term will bring higher inflation. This series of contradictory policies, first to increase economic growth and secondly to increase the level of inflation, in a capitalist economy is referred to as "The paradox capitalist"

## AN ALTERNATIVE SOLUTION: THE SHAPLEY SOLUTION

Assume that players agree to cooperate. The distribution solution thus obtained depends on the properties proposed. We study the results of the Shapley solution. Static cooperative games are studied by defining a characteristic function v, which associates each coalition of players to coordinate their strategies in this dynamic version of the characteristic function assigned to each operator will be the payment received from his maximin strategy. The Shapley value for the case of two players is as follows:

$$\begin{cases} \phi_1 = \frac{1}{2} \left[ \upsilon(\{1\}) + \upsilon(\{1,2\}) - \upsilon(\{2\}) \right] \\ \phi_2 = \frac{1}{2} \left[ \upsilon(\{2\}) + \upsilon(\{1,2\}) - \upsilon(\{1\}) \right] \end{cases}$$
(24)

Where:

$$\upsilon(\{1\}) = J_1(\alpha^*(t), \beta^*(t)) = \frac{b\theta AK_0}{\lambda}$$
$$\upsilon(\{2\}) = J_2(\alpha^*(t), \beta^*(t)) = \frac{(1-b)\theta AK_0}{\Psi}$$

And considering the Pareto solution of the problem (8) into cooperative solution is:

$$\upsilon(\{1,2\}) = \frac{\psi a \theta A K_0 + (\lambda - (1 - a)\theta A)((1 - a)\theta A K_0)}{(\lambda - (1 - a)\theta A)\psi}$$

Now, we find the value of cooperative mapping between the two players. For the agent "Consumer worker", the Shapley value is:

$$\phi_1 = \frac{1}{2} \Bigg[ \frac{\theta A K_0 \left[ \psi (b + \lambda a) + \lambda (\lambda - (1 - a) \theta A) (b - a) \right]}{\lambda \psi (\lambda - (1 - a) \theta A)} \Bigg]$$

And, for the agent "Consumer investor";

$$\phi_{2} = \frac{1}{2} \left[ \frac{\theta A K_{0} \left[ \lambda (1 - b + \psi) + (\lambda - (1 - a)\theta A) (\lambda (1 - a) - \psi b) \right]}{\lambda \psi (\lambda - (1 - a)\theta A)} \right]$$

It is easy to realize that the payments received in cooperation between agents are much larger than available in a centralized and decentralized economy.

## CONCLUSION

The innovations of the Lancaster model involving the existence of different discount rates for each of the players and the introduction of an institutional framework that annotates player control, significantly enriches the analysis of differential game of economic growth. We can say that equilibrium of the game found from the perspective of a centralized economy, are established in the value of redistribution between agents, by the parameters ( $A, \lambda, \psi, \theta$ ) and the values set in the political process dimension to the discretion of the players control (a, b, c, d) and the Nash solution obtained is an optimal bang-bang type, optimum control for the capitalists of c, which is interpreted as the minimum permissible capitalization which must govern the social planner to maximize consumption in the economy. It determines a Pareto optimal set of all possible payments in the game, prompted by the social planner and establishes controls that reach, denoting two stages in this process of capital stock, accumulation and maximum consumption.

With respect to the dynamics of the model, we obtained positive growth rates and constant over time and phase paths of the model show a globally unstable system, but show a steady growth in consumption with increases in the capital stock. It is the solution based on decentralization, in which each player's optimal controls are reversed to those found in the centralized economy, where the player gets a capitalist optimum control variable value at its maximum dimension (higher levels of capitalization ), while player worker obtains optimal control variable at its minimum value of dimension (extremely low consumption values). And, unlike the centralized economy, steady states are obtained in the economy, and rates of discount factor equal to the rate of inflation. From this perspective, the economy will have high interest rates and the dynamics of the model, leads to continued increases in the level of wages ending with higher inflation rates. It is evident that the value of redistribution of consumption in the decentralized economy is greater than from a centralized economy.

Economic policy can be adopted in a decentralized economy, showing that inflation is determined under the econometric equation proposed in this research, and that the rate of economic growth reaches a steady state if the population is growing greater than the interest rates of the economy. It was shown that increasing the level of wages results in short run higher inflation. Economies that have no monetary support are advised not to increase the level of wages unconsciously.

We also conducted an analysis of a cooperative solution between agents (Shapley value), where these covenants and agreements of the game both reach maximum levels of consumption. This solution proved that the rate of economic growth will be negative and the value of redistribution among agents is greater than available in a centralized and decentralized economy.

## REFERENCES

Buhl, H.U and W. Machaczek (1985). Workers' investment and return in a differential game model.

De Zeeuw, A. (1992). Note on "Nash and Stackelberg solutions in a differential game model of capitalism". *Journal of Economic Dynamics and Control*, N° 16, pp. 139-145.

Gonzales, C. (1998). Economic growth models: contributions from the theory of dynamic games. Doctoral Thesis. Universidad de la Laguna.

Hoel, M. (1978). Distribution and growth as a differential game between workers and capitalists. *International Economic Review*, N° 19, pp. 335-350.

Lancaster, K. (1973). The Dynamic Inefficiency of Capitalism. *Journal of Political Economy*, Vol. 81, pp. 1092-1109.

Nash, J. (1951). Non-Cooperative Games. Annals of Mathematics, Vol. 54, pp. 286-295.

Pohjola, M. (1983a). Nash and Stackelberg solutions in a differential game model of capitalism, *Journal of Economics Dynamics and Control* N° 6, pp. 173-186.

Pohjola, M. (1983b). Workers's investment funds and the dynamic inefficiency of capitalism, Nash and Stackelberg solutions in a differential game model of capitalism, *Journal of Public Economics* N° 20, pp. 271-279.

Pohjola, M. (1985). Growth, Distribution and Employment Modelled as a Differential Game. *Optimal Control Theory and Economic Analysis 2*. pp. 265-290.

Ramsey, F. (1928). A Mathematical Theory of Savings. *Economic Journal*, Vol. 38, n° 152, pp. 543-559. Rebelo, S. (1991). Long-run policy analysis and long-run growth. *Journal of Political Economy*, N° 99, pp. 500-521.

Rincon, J. (1993). Shapley value in a capitalist game. *Annals of Economic and Business Studies*, No. 8, pp. 129-138

Rincon, J.P.(1994). Some cooperative solutions differential applied to a game. Meeting on Game Theory, Bilbao 14-16 September.

Seierstad, A. (1993). The dynamic efficiency of capitalism, *Journal of Economic Dynamics and Control*, N° 17, pp. 877-886.

Sidrauski, M. (1967). Rational choice and patterns of growth in a monetary economy. *American Economic Review Papers and Proceedings*, N° 57, pp. 534–544.

Soto, M. (1994). Distribution, Growth and Employment. A non-cooperative solution. *Applied Economic Studies*, Vol II, pp. 39-46.

Soto, M. and Fernandez, R. (1992). Optimal strategies in the game with both Lancaster differential update. *Annals of Economic and Business Studies*, N°. 7, pp. 207-215.

Soto, M. and Fernandez, R. (1995). Stackelberg strategies in a differential game. *Annals of Economic and Business Studies*. N ° 10, pp. 157-166

Von Neumann, J. y Morgenster, O. (1944). *Theory of Games and Economic Behavior*. *John Wiley & Sons, Inc.,* New York.

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